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Two-dimensional flow under gravity in a jet of viscous liquid

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This paper is concerned with the steady, symmetric, two-dimensional flow of a viscous, incompressible fluid issuing from an orifice and falling freely under gravity. A Reynolds number is defined and considered to be small. Due to the apparent intractability of the problem in the neighbourhood of the orifice, interest is confined to the flow region below the orifice, where the jet is bounded by two free streamlines. It is assumed that the influence of the orifice conditions will decay exponentially, and so the asymptotic solutions sought have no dependence upon the nature of the flow at the orifice. In the region just downstream of the orifice, it is expected that the inertia effects will be of secondary importance. Accordingly the Stokes solution is sought and a perturbation scheme is developed from it to take account of the inertia effects. It was found possible only to express the Stokes solution and its perturbations in the form of co-ordinate expansions. This perturbation scheme is found to be singular far downstream due to the increasing importance of the inertia effects. Far downstream the jet is expected to be very thin and the velocity and stress variations across it to be small. These assumptions are used as a basis in deriving an asymptotic expansion for small Reynolds numbers, which is valid far downstream. This expansion also has the appearance of being valid *very* far downstream, even for Reynolds numbers which are not necessarily small. The method of matched asymptotic expansions is used to link the asymptotic solutions in the two regions. An extension of the method deriving the expansion far downstream, to cover the case of an axially-symmetric jet, is given in an appendix.

1. Introduction

An incompressible viscous fluid passes through a two-dimensional orifice and then falls vertically and symmetrically (see figure 1). The fluid region thereafter is bounded by two free streamlines, and the medium outside this region is assumed to be at zero pressure and not to interact with the jet. The effects of surface tension are ignored. Gravity will accelerate the fluid and so, by continuity, there will be a contraction of the jet. This contraction will give rise to viscous stresses within the jet which will in turn produce an effect upon the velocity field. Eventually we expect the jet to be extremely thin and each fluid particle to be falling as a solid body, i.e. with the inertia effects dominating the viscous effects.

We would, of course, have liked to solve the problem in the whole of the fluid region. It was formulated and considered in some detail, but appeared to be intractable, even for Stokes flow, because of the difficulty arising from the mixed non-linear boundary conditions (the unknown function $z(t)$ describing the boundary enters these conditions in a non-linear way). In this paper we restrict our attention to the region of the flow below the orifice.

In the region close to the orifice it is expected that the viscous effects will dominate the inertia effects. In similar problems, the strongly diffusive nature of such effects invariably indicates that the influence of the conditions at the orifice will decay exponentially fast downstream (the orifice conditions referred to are the fine characteristics such as the variation of the velocity and stresses across the flow, rather than bulk characteristics such as total mass flux). The rate of this decay is also usually so fast that such an influence is insignificant at a distance, from the orifice, of order of the orifice width. However, in this particular problem it appears to be difficult to justify such a decay analytically. This is because vorticity is being produced at the free streamlines and the difficulty lies in estimating, *a priori*, the rate of this production. These statements are justified in §3.1. It does, however, remain extremely plausible that such a decay does in fact occur and we shall accordingly assume that this is so. This means that we shall seek solutions, away from the orifice, which are independent of the orifice conditions.

By considering the flow only in a region away from the orifice, we have eliminated the natural length scale of the orifice width. It follows that the only parameters appearing in the problem are Q, g and ν, where $2Q$ is the volume flux across (unit depth of) any section of the jet, g is the acceleration due to gravity and ν is the kinematic viscosity. Therefore the only dimensionless parameter is Q/ν, this we define to be the Reynolds number and denote it by R. In what follows we consider R to be small.

In order to emphasize the subdominance of the inertia terms in the region just below the orifice, we take as basic length and velocity scales, in this region, $(\nu Q g^{-1})^{\frac{1}{3}}$ and $(gQ^2\nu^{-1})^{\frac{2}{3}}$. If we express the field equations in terms of variables made dimensionless with respect to these scales, and employ a perturbation scheme based upon the smallness of R (the basic solution being obtained by solving the problem for $R = 0$), then the solution will be incorrect far downstream. A solution which will be correct far downstream must take the inertia terms into account from the outset, as in this region these terms are comparable and eventually dominate the viscous terms. The simplifying features of this region are that the jet will be thin and the velocity and stress gradients across it will be finite. These features are brought to the fore by the use of $(\nu^2 g^{-1})^{\frac{1}{3}}$ and $(\nu g)^{\frac{1}{3}}$ as length and velocity scales in this region. An expansion is derived, on these assumptions, and is then matched with the 'inner' expansion. The term 'outer' expansion refers to the expansion which is valid far downstream, and its region of validity is termed the 'outer' region. Similarly the inner expansion is valid in the inner region, which is upstream of the outer region.

The two main difficulties encountered in the determination of viscous flows with free streamlines are that these streamlines are not known in advance and must be found as part of the solution, and that the boundary conditions to be imposed upon these unknown boundaries are stress conditions, which themselves depend upon the unknown orientation of the streamlines. In the inner expansion the former difficulty is overcome by successive approximations of the boundary curve, whilst the latter difficulty is overcome by the introduction of a stress function Φ, and this in turn suggests that the Navier–Stokes equations should be rewritten in a complex form. This form is developed in appendix A. It was first presented by Legendre (1949), without any derivation, and was used by him to discuss the flow over a flat plate. Moisil (1955), and following him Langlois (1964), derive the reformulated field equations (and boundary conditions applicable to free streamlines) for the case of Stokes flow, though the motivation and procedure appear to be quite different from the one presented in appendix A. In a recent paper Garabedian (1966) uses the complex variable form of the equations for Stokes flows with free streamlines in a variety of idealized situations. Unfortunately the complexity of the present problem has permitted us to derive only a co-ordinate expansion (i.e. for large values of the streamwise co-ordinate) of the inner problem. This is, however, sufficient to provide the missing boundary conditions for the outer expansion.

In the outer expansion both difficulties are overcome by expressing the field equations and boundary conditions in flowline co-ordinates, that is, a system of co-ordinates in which one of the independent variables is the stream function and the other variable is so constructed as to form an orthogonal net.

Strictly speaking, the two expansions should be developed side by side and matched at each stage before proceeding to the subsequent stage. However, in view of the fact that the two expansions are derived by such different methods, several terms of the inner expansion are derived here before the outer expansion is considered. Where a step in one expansion is dependent upon the previous stage in the other expansion, this will be noted and explained in the text.

Brown (1961) gave details of some experimental work on viscous sheets, and in an appendix to that paper, Taylor gives a derivation of the equation of motion of a one-dimensional viscous jet falling under gravity. This equation is the same as the one we derive for the leading term of our outer expansion, though the methods of derivation are dissimilar.

In appendix B, we derive an outer expansion for the case of an axially symmetric jet. This is included because we may use essentially the same technique as is used for the two-dimensional case, and also because the ordinary differential equations involved in the solution may be reduced to those occurring in the two-dimensional flow.

2. The inner expansion

2.1. *Formulation*

We take as the origin of a rectangular cartesian co-ordinate system a point on the line of symmetry at an arbitrary distance below the orifice. The X-axis is taken to lie in the direction in which gravity acts (i.e. along the line of symmetry), and the Y-axis as in figure 1. The components of the velocity in the X and Y

directions are denoted by U and V. Dimensionless variables are introduced as follows,

$$\left.\begin{aligned}(X, Y) = (\nu Q g^{-1})^{\frac{1}{3}}(x, y), \quad (U, V) = (g Q^2 \nu^{-1})^{\frac{1}{3}}(u, v),\\ P = \rho g (\nu Q g^{-1})^{\frac{1}{3}} p, \quad \Phi = 2\rho\nu Q\phi, \quad \Psi = Q\psi,\end{aligned}\right\} \tag{2.1}$$

where Φ and Ψ are the dimensional Airy stress function and stream function, introduced in appendix A, P is the pressure defined so that P vanishes outside

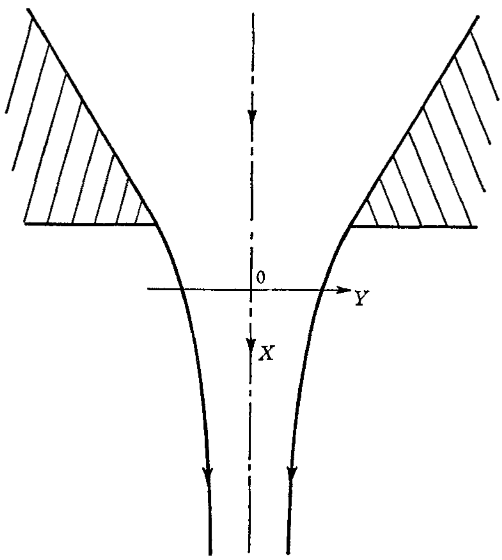

FIGURE 1. The flow region.

the jet, and ρ is the density assumed constant throughout. We define, as the only dimensionless parameter appearing in the problem, a Reynolds number $R = Q/\nu$ and consider this to be small. The reader is now referred to appendix A, where the governing equations and the boundary conditions are expressed in complex variable form. These reformulations are, in terms of the dimensionless variables,

$$\frac{\partial^2}{\partial z^2}(\phi + i\psi) = -\tfrac{1}{2}R\left(\frac{\partial\psi}{\partial z}\right)^2, \tag{2.2}$$

together with the boundary conditions

$$\frac{\partial\phi}{\partial z} = \frac{1}{4}\int x(1 - iS'(x; R))\,dx \tag{2.3}$$

and

$$\operatorname{Re}\left\{\frac{\partial\psi}{\partial z}(1 + iS'(x; R))\right\} = 0, \tag{2.4}$$

which are to be applied on the unknown free streamlines denoted by $y = \pm S(x; R)$. Here $S'(x; R)$ denotes the derivative of S with respect to x. Equation (2.2) is the amalgamation of the Navier–Stokes equations and the equation of continuity, equation (2.3) is the condition stating that the normal and shear stresses must vanish on the free streamlines and (2.4) asserts that there is to be no normal velocity on $y = \pm S(x; R)$.

2.2. *The zero-order approximation*

The dependent variables ϕ and ψ are functions of $z, \bar{z}$ and R. The assumption is now made that they may be expanded in a series in functions of R, and that the first term in each such series will be independent of R.

The zero-order field equations may be obtained formally by putting $R = 0$ in (2.2), to obtain

$$\frac{\partial^2}{\partial \bar{z}^2}(\phi_0 + i\psi_0) = 0. \tag{2.5}$$

This is, of course, the Stokes approximation and the solution of (2.5) will be the Stokes flow. The general solution of (2.5) is

$$\phi_0 + i\psi_0 = \bar{z}\overline{f_0(z)} + \overline{g_0(z)}, \tag{2.6}$$

where $f_0(z)$ and $g_0(z)$ are analytic functions of z. We now wish to express the boundary conditions in terms of f_0 and g_0. We differentiate (2.6) with respect to z and with respect to $\bar{z}$ and take the complex conjugate of the latter resulting equation, to derive

$$\frac{\partial \phi_0}{\partial \bar{z}} + i\frac{\partial \psi_0}{\partial \bar{z}} = \overline{f_0(z)},$$

$$\frac{\partial \phi_0}{\partial z} - i\frac{\partial \psi_0}{\partial z} = \bar{z}f_0'(z) + g_0'(z).$$

The sum and difference of these two equations gives

$$\frac{\partial \phi_0}{\partial z} = \tfrac{1}{2}\{\overline{f_0(z)} + \bar{z}f_0'(z) + g_0'(z)\}, \tag{2.7}$$

$$\frac{\partial \psi_0}{\partial z} = -\frac{i}{2}\{\overline{f_0(z)} - \bar{z}f_0'(z) - g_0'(z)\}. \tag{2.8}$$

We substitute (2.7) and (2.8) into the boundary conditions (2.3) and (2.4), to derive the following equations which are to be satisfied on $y = \pm S_0(x)$,

$$f_0(z) + z\overline{f_0'(z)} + \overline{g_0'(z)} = \tfrac{1}{4}x^2 + \frac{i}{2}\int xS_0'(x)\,dx, \tag{2.9}$$

$$\operatorname{Im}\{(1 - iS_0'(x))(f_0(z) - z\overline{f_0'(z)} - \overline{g_0'(z)})\} = 0. \tag{2.10}$$

It can be seen that (2.9) and (2.10) represent a boundary-value problem in analytic function theory. Such problems are familiar in the theory of plane elasticity. In our case there is the added difficulty that $S_0(x)$ is also unknown and must be found from (2.9) and (2.10), and as such (2.10) is clearly of a non-linear character. At present it has only been possible to derive an asymptotic solution (for large x) to this problem.

If a solution to (2.9) and (2.10) exists, then it is clear that for large values of x it must have the asymptotic form

$$f_0(z) = O(z^2), \quad g_0(z) = O(z^3), \quad S_0(x) = O(x^{-2}).$$

The last of these, though not at first obvious, may be obtained by putting $S_0(x) = O(x^{-n})$ in (2.10). Hence we put

$$f_0(z) = a_0 z^2 + o(z^2),$$
$$g_0(z) = b_0 z^3 + o(z^3),$$
$$S_0(x) = c_0 x^{-2} + o(x^{-2}),$$

and insert these into (2.9) and (2.10) to find $a_0 = \frac{3}{16}$ and $b_0 = -\frac{5}{48}$. c_0 remains undetermined as (2.10) and the imaginary part of (2.9), from which we would expect to find c_0, are both homogeneous in c_0. However, we require that the volume flux across any section should equal 2. Therefore

$$\int_0^{S_0(x)} u_0(x, y)\, dy = 1, \tag{2.11}$$

to the first order in R. From (2.8) we see that

$$u_0 + iv_0 = f_0(z) - z\overline{f_0'(z)} - \overline{g_0'(z)}, \tag{2.12}$$

and so from (2.11) and (2.12) we find that $c_0 = 8$.

The error involved in the omission of terms smaller than those considered is of order x^{-6} times the retained terms, and so for a second approximation we put

$$f_0(z) = \tfrac{3}{16} z^2 + a_{01} z^{-4} + o(z^{-4}),$$
$$g_0(z) = -\tfrac{5}{48} z^3 + b_{01} z^{-3} + o(z^{-3}),$$
$$S_0(x) = 8x^{-2} + c_{01} x^{-8} + o(x^{-8}).$$

Again we insert these expressions into (2.9) and (2.10) to find that $a_{01} = -\frac{38}{5}$, $b_{01} = \frac{274}{15}$ and $c_{01} = \frac{1792}{15}$. The error involved is again of order x^{-6} times the smallest retained term. Further terms may be calculated in the same way.

If it is attempted to introduce into the solution intermediate terms, say αz in $f_0(z)$, and a corresponding chain of terms in $g_0(z)$ and $S_0(x)$ and the induced lower order terms in $f_0(z)$, then this succeeds and α remains arbitrary, other than that it should be real. However, when any physical quantities, such as the velocity, are derived from these solutions, they appear in a form whose natural variable is $(z + 4\alpha)$. This shows these parts of the solution arise merely because of the arbitrariness of our origin. A suitable redefinition of the origin will simply remove the terms dependent upon α. This we will do by formally setting $\alpha = 0$. There is no loss of generality in this procedure as we are replacing one arbitrary origin by another. It should be noted that the absence of any arbitrariness in our solution, other than of the type mentioned above, is in complete agreement with the assumption that the influence of the orifice conditions decays exponentially fast. This is because any such decaying terms would automatically be excluded from our asymptotic solution.

The solution to the zero-order problem, written as an expansion for large x, is

$$\left.\begin{aligned} f_0(z) &= \tfrac{3}{16} z^2 - \tfrac{38}{5} z^{-4} + O(z^{-10}), \\ g_0(z) &= -\tfrac{5}{48} z^3 + \tfrac{274}{15} z^{-3} + O(z^{-9}), \\ S_0(x) &= 8x^{-2} + \tfrac{1792}{15} x^{-8} + O(x^{-14}). \end{aligned}\right\} \tag{2.13}$$

When we match the inner and outer expansions we require a form of u_0 (for large x) on the line of symmetry (the matching of all other expressions is automatically accomplished when we match u on $y = 0$). From (2.12) and (2.13) we find that this form is

$$u_0(x,0) = \tfrac{1}{8}x^2 + \tfrac{84}{5}x^{-4} + O(x^{-10}). \tag{2.14}$$

2.3. *The first-order approximation*

At this stage we could calculate the dominant term of the outer expansion, matching it to (2.14) in order to determine an arbitrary constant in the outer expansion. It would then appear that, if the two expansions are to match to higher orders, the inner expansions for the pressure and the velocity vector $\mathbf{u}$ must take the form

$$\left.\begin{aligned} p &= p_0 + R^{\frac{1}{3}}p_1 + R^{\frac{2}{3}}p_2 + Rp_3 + \ldots, \\ \mathbf{u} &= \mathbf{u}_0 + R^{\frac{1}{3}}\mathbf{u}_1 + R^{\frac{2}{3}}\mathbf{u}_2 + R\mathbf{u}_3 + \ldots. \end{aligned}\right\} \tag{2.15}$$

Consequently ϕ, ψ and S will also be represented in series in powers of $R^{\frac{1}{3}}$. It is only the matching that can force the existence of ϕ_1, ψ_1 and S_1 and, as we shall see, these terms vanish identically. We shall therefore omit them at this stage. With $\mathbf{u}_1$ and p_1 absent, it is only the matching that forces the existence of $\mathbf{u}_2$ and p_2. These terms are present as we shall justify in §4.

2.4. *The second-order approximation*

The field equations for ϕ_2 and ψ_2 are still homogeneous,

$$\frac{\partial^2}{\partial \bar{z}^2}(\phi_2 + i\psi_2) = 0,$$

and hence

$$\phi_2 + i\psi_2 = z\overline{f_2(z)} + \overline{g_2(z)}.$$

The boundary will experience a shift of order $R^{\frac{2}{3}}$, and so in the non-linear boundary conditions we must also include the zero-order terms as they will also contribute terms of order $R^{\frac{2}{3}}$, through the boundary shift. The total boundary conditions, to this order in R, are

$$\left.\begin{aligned} (f_0 + z\overline{f_0'} + \overline{g_0'}) + R^{\frac{2}{3}}(f_2 + z\overline{f_2'} + \overline{g_2'}) &= \tfrac{1}{4}x^2 + \frac{i}{2}\int x(S_0' + R^{\frac{2}{3}}S_2')\,dx, \\ \operatorname{Im}\{[1 - i(S_0' + R^{\frac{2}{3}}S_2')][(f_0 - z\overline{f_0'} - \overline{g_0'}) + R^{\frac{2}{3}}(f_2 - z\overline{f_2'} - \overline{g_2'})]\} &= 0, \end{aligned}\right\} \tag{2.16}$$

on

$$y = \pm(S_0(x) + R^{\frac{2}{3}}S_2(x)).$$

We use hints from the matching and look for solutions of the form

$$f_2(z) = a_2 z^4 + o(z^4),$$
$$g_2(z) = b_2 z^5 + o(z^5),$$
$$S_2(x) = c_2 + o(1).$$

We substitute these expressions into (2.16) to find that

$$b_2 = -a_2 \quad \text{and} \quad c_2 = -128a_2,$$

where a_2 is real but otherwise arbitrary. The terms, smaller in z, may be found in the same way as in the zero-order approximation. For matching purposes, the second-order contribution to u on the line of symmetry is

$$R^{\frac{2}{3}}(2a_2 x^4 + O(x^{-2})).$$

2.5. *The third-order approximation*

Here the inertia terms make their first appearance. From appendix A, the field equations are

$$\frac{\partial^2}{\partial z^2}(\phi_3 + i\psi_3) = \tfrac{1}{8}(u_0 - iv_0)^2 = 2^{-11}(5z^2 - 6z\bar{z} + 3\bar{z}^2)^2 + O(z^{-2}). \tag{2.17}$$

We have retained only the leading terms of $(u_0 - iv_0)$ as only these will contribute to the leading terms of $\phi_3 + i\psi_3$. The general solution of (2.17) is

$$\phi_3 + i\psi_3 = z\overline{f_3(z)} + \overline{g_3(z)} + 2^{-11}(\tfrac{5}{6}z^6 - 3z^5\bar{z} + \tfrac{11}{2}z^4\bar{z}^2 - 6z^3\bar{z}^3 + \tfrac{9}{2}z^2\bar{z}^4) + O(\log z).$$

We could proceed in the same way as for the second-order approximation, but in view of the increasing complexity only the results will be stated. The solutions will clearly be forced by the inertia terms, which indicates the order of the leading terms. The solution is given by

$$\begin{aligned} f_3(z) &= -2^{-11}z^5 + O(z^{-1}), \\ g_3(z) &= -\tfrac{5}{6}2^{-11}z^6 + O(1), \\ S_3(x) &= -\tfrac{1}{8} + O(x^{-6}). \end{aligned}$$

The leading contribution to u on the line of symmetry is $R2^{-9}x^5$.

We could, in principle, pursue our calculations further, but in view of the similarity of method and the absence of any salient results we shall not do so here. It will be noticed that at each approximation there is a singularity at $z = \infty$ and that the singularity becomes worse as we raise the order of approximation. The solution as given by the inner expansion will therefore be incorrect far downstream as the perturbation is singular at downstream infinity.

3. The outer expansion

3.1. *Formulation*

Far downstream we expect the jet to become very thin and the variations in the velocity and stress across it to become small. In order to utilize this we define dimensionless variables as follows,

$$\left.\begin{aligned} (X, Y) = (\nu^2 g^{-1})^{\frac{1}{3}}(\hat{x}, \hat{y}), \quad (U, V) = (\nu g)^{\frac{1}{3}}(\hat{u}, \hat{v}), \\ P = \rho(\nu g)^{\frac{2}{3}}\hat{p}, \quad \Psi = \nu R\psi, \end{aligned}\right\} \tag{3.1}$$

where the quantities denoted by capital letters are as in the inner problem. Ψ, the dimensional stream function, is made dimensionless as in (3.1) so that the free streamlines are denoted by $\psi = \pm 1$, as they were in the inner problem. $R = Q/\nu$ is the Reynolds number as defined in the inner problem. The relationship between the inner and outer variables is given by

$$(x, y) = R^{-\frac{1}{3}}(\hat{x}, \hat{y}), \quad (u, v) = R^{-\frac{2}{3}}(\hat{u}, \hat{v}), \quad p = R^{-\frac{1}{3}}\hat{p}.$$

The way in which we make Y dimensionless does not, of course, make $\hat{y}$ of order unity in the region considered (unlike $\hat{x}$), but this is immaterial as we shall be treating $\hat{y}$ as a dependent variable in what follows. We shall now omit the symbol ($\hat{\ }$) for convenience, and restore it when we match the two expansions formally.

As before, the free streamlines are unknown in terms of x and y, but are given by $\psi = \pm 1$. This suggests using ψ as an independent variable. We therefore consider the problem to be in the ζ-plane, where $\zeta = (\xi, R\psi)$. Here ξ is defined by $\xi = x$ on the line of symmetry, and the lines $\xi =$ constant are everywhere orthogonal to the lines $R\psi =$ constant. This means that ξ and $R\psi$ form an orthogonal curvilinear co-ordinate system.

If we put $q^2 = u^2+v^2$, then the velocity components with respect to $(\xi, R\psi)$ are $(q, 0)$, whereas those with respect to (x,y) were $(u,v) = (q\cos\theta, -q\sin\theta)$. This defines the angle θ. The two planes are linked by the following transformation,

$$x = \xi + R\int_0^{\psi} q^{-1}\sin\theta\, d\psi, \tag{3.2}$$

$$y = R\int_0^{\psi} q^{-1}\cos\theta\, d\psi. \tag{3.3}$$

The arc-length parameter associated with $R\psi$ is q^{-1}, and that with ξ we denote by h. The formal definition of h is $h^2 = (\partial x/\partial \xi)^2 + (\partial y/\partial\xi)^2$, and so h could be derived in terms of q and θ from (3.2) and (3.3). This is, however, an arduous task and not very illuminating and so we adopt an alternative approach. Consider the constant vector grad x, which in this problem is representative of gravity. In the ζ-plane

$$\operatorname{grad} x = \left(\frac{1}{h}\frac{\partial x}{\partial \xi}, \frac{q}{R}\frac{\partial x}{\partial \psi}\right) = \left(\frac{1}{h}\frac{\partial x}{\partial \xi}, \sin\theta\right),$$

on using equation (3.3). As the magnitude of grad x is unity, we have that $(1/h)/(\partial x/\partial\xi) = +\cos\theta$, the positive sign being implied by (3.2). Therefore

$$\operatorname{grad} x = (\cos\theta, \sin\theta).$$

We know that curl (grad x) and div(grad x) both vanish identically and so expressing this fact in the ζ-plane, we have

$$\frac{\partial}{\partial\xi}(q^{-1}\cos\theta) + R^{-1}\frac{\partial}{\partial\psi}(h\sin\theta) = 0, \tag{3.4}$$

$$\frac{\partial}{\partial\xi}(q^{-1}\sin\theta) - R^{-1}\frac{\partial}{\partial\psi}(h\cos\theta) = 0. \tag{3.5}$$

From these equations we may deduce, quite simply, that

$$h = Rq_{\xi}/(q^2\theta_{\psi}), \tag{3.6}$$

$$h_{\psi} = R\theta_{\xi}/q, \tag{3.7}$$

where the subscripts denote differentiation with respect to the variable indicated.

Having established the geometry of the ζ-plane, we may express the Navier–Stokes equations with respect to these co-ordinates,

$$q\frac{\partial q}{\partial \xi}+\frac{\partial p}{\partial \xi} = h\cos\theta + R^{-2}qh\frac{\partial}{\partial \psi}\left\{\frac{q}{h}\frac{\partial}{\partial \psi}(qh)\right\}, \tag{3.8}$$

$$-\frac{q^2}{h}\frac{\partial h}{\partial \psi}+\frac{\partial p}{\partial \psi} = \frac{R\sin\theta}{q} - \frac{1}{qh}\frac{\partial}{\partial \xi}\left\{\frac{q}{h}\frac{\partial}{\partial \psi}(qh)\right\}. \tag{3.9}$$

The equation of continuity is automatically satisfied by the use we have made of the stream function as an independent variable. We have the four field equations (3.6)–(3.9) for the four unknowns q, h, p and θ.

The boundary conditions are particularly simple in the ζ-plane. The normal velocity condition is automatically satisfied by the use we have made of ψ, and from the stress tensor we calculate that the zero shear and normal stress conditions, to be applied on $\psi = \pm 1$, are

$$R^{-1}qh\frac{\partial}{\partial \psi}\frac{q}{h} = 0, \tag{3.10}$$

$$p = -\frac{2}{h}\frac{\partial q}{\partial \xi}. \tag{3.11}$$

Because we shall not be considering the region in which q could vanish on the free streamlines, and as h is non-zero, (3.10) may be written as

$$\frac{\partial}{\partial \psi}\frac{q}{h} = 0. \tag{3.12}$$

We can now see that by expressing the problem in the ζ-plane we have overcome the difficulties of the unknown boundary and the stress conditions to be applied there. By way of compensation, the field equations are now more complicated. It is this complication which makes this formulation unsuitable for the problem in the inner region.

We shall digress at this point to comment upon the remarks made in §1 concerning the analytic justification of an exponentially fast decay in the influence of the orifice conditions. First, we note that (3.8) may be written in the form

$$R^{-1}q\frac{\partial \omega}{\partial \psi} = -\frac{1}{h}\frac{\partial H}{\partial \xi}, \tag{3.13}$$

where $\omega = -R^{-1}(q/h)(\partial/\partial\psi)(qh)$ is the vorticity and $H = \frac{1}{2}q^2+p+\Omega_*$ is the total head, Ω_* being the potential of the force due to gravity. From this form of the equation we can see that vorticity will be transported across a streamline whenever the total head is changing along it. In particular, along a free streamline the total head will, in general, change (unlike the case of an inviscid fluid), and so vorticity will in general be produced on a free streamline. We are ignorant of the shape of the free streamlines and so we are unable to derive an *a priori* estimate for the rate of production of this vorticity and its dependence upon the orifice conditions. It is the possibility of introducing, from free streamlines, some non-exponentially decaying influence of the orifice conditions which prevents the usual estimation analysis from being effected. An alternative analysis has so far eluded the author. We are therefore forced to assume the plausible type of decay rate.

Returning to the problem in the outer region, it can be seen from symmetry considerations that q, h and p will be even functions of ψ, and θ will be an odd function of ψ. In the field equations, (3.6)–(3.9), R always appears in conjunction with ψ and so we suppose that the solutions may be expressed in the forms

$$\left.\begin{aligned} q &= q_0 + R^2\psi^2 q_2 + R^4\psi^4 q_4 + \dots, \\ h &= 1 + R^2\psi^2 h_2 + R^4\psi^4 h_4 + \dots, \\ p &= p_0 + R^2\psi^2 p_2 + R^4\psi^4 p_4 + \dots, \\ \theta &= R\psi\theta_1 + R^3\psi^3\theta_3 + \dots. \end{aligned}\right\} \tag{3.14}$$

Although R and ψ appear together in the field equations, R does not appear explicitly with ψ in the designation of the boundary. We therefore assert that the coefficients of $(R\psi)$ in (3.14) will be functions of both ξ and R. When we match the inner and outer expansions, it becomes apparent that the coefficients must have the form, to take a typical example,

$$q_0(\xi; R) = q_{00}(\xi) + R^2 q_{02}(\xi) + R^4 q_{04}(\xi) + \dots. \tag{3.15}$$

3.2. *The derivation and solutions of the equations for q_{00}, etc.*

We insert the forms (3.14) and (3.15) *et al.* into the equations (3.6) and (3.7) and also into the boundary conditions (3.11) and (3.12), to obtain relationships between the various coefficients appearing in (3.14) and (3.15). From the expressions arising from (3.6) and (3.7) we compare terms independent of R to obtain, on some rearrangement,

$$\theta_{10} = q'_{00}/q_{00}^2, \tag{3.16}$$

$$h_{20} = \tfrac{1}{2}q''_{00}/q_{00}^3 - q'^2_{00}/q_{00}^4. \tag{3.17}$$

Here ($'$) denotes differentiation with respect to ξ. Similarly the leading terms in (3.11) and (3.12) give

$$p_{00} = -2q'_{00} \tag{3.18}$$

and

$$q_{20} = q_{00}h_{20}, \tag{3.19}$$

respectively. We now substitute (3.14) and (3.15) into (3.8) and equate the terms independent of R to zero. In the resulting equation we use (3.16)–(3.19) to derive a second-order non-linear ordinary differential equation for q_{00},

$$4q''_{00} - 4q'^2_{00}/q_{00} - q_{00}q'_{00} + 1 = 0. \tag{3.20}$$

As may be readily verified by reverting to dimensional variables, the last term in this equation is a gravity term, the next last is the inertia term and the first two are essentially viscous terms (the first term in fact arises equally from the viscous stresses and the pressure forces, though these forces rely largely upon the viscous effects for their importance in this region). It should be noted from this that viscosity remains a relevant factor in the outer region.

We may derive the equations for q_{02} by taking account of terms of order R^2 and $R^2\psi^2$ in the field equations and of terms of order R^2 in the boundary conditions. This is a process of considerable manipulative complexity and so only the resulting equation will be quoted,

$$4q''_{02} - (q_{00} + 8q'_{00}/q_{00})q'_{02} + (4q'^2_{00}/q^2_{00} - q'_{00})q_{02} = M(\xi), \tag{3.21}$$

where M may be expressed in the form

$$M(\xi) = \frac{16}{3}\frac{q'^4_{00}}{q^5_{00}} - 3\frac{q'^3_{00}}{q^3_{00}} + \frac{1}{2}\frac{q'^2_{00}}{q_{00}} - \frac{1}{48}q_{00}q'_{00} - \frac{13}{12}\frac{q'_{00}}{q_{00}} + \frac{1}{48} + 7\frac{q'^2_{00}}{q^4_{00}} + \frac{5}{12}\frac{1}{q^3_{00}}.$$

In principle the higher-order terms q_{0n} could be calculated in a similar manner, though the labour involved would be prohibitive. It is clear, however, that the q_{0n} will satisfy an equation with the same homogeneous part as that in (3.21), all the q_{0n} being perturbations developed from q_{00}.

It is apparent from (3.16)–(3.19) and their higher-order counterparts that the q_{0n} completely determine all the other coefficients in (3.14) and (3.15) *et al.*, and so by solving (3.20) and (3.21), etc., we shall, in principle, have solved the outer problem to any desired order in R.

First, we make a slight transformation of the variables in order to simplify the arithmetic. We put $\xi = 2^{\frac{4}{3}}\sigma$ and $q_{0n} = 2^{\frac{2}{3}}F_n$. In terms of these variables, equation (3.20) becomes

$$F''_0 - F'^2_0/F_0 - F_0F'_0 + 1 = 0, \tag{3.22}$$

where ($'$) now denotes differentiation with respect to σ. This is the form in which Taylor (see Brown (1961)) expressed his equation. Similarly equation (3.21) becomes

$$F''_2 - (F_0 + 2F'_0/F_0)F'_2 + (F'^2_0/F^2_0 - F'_0)F_2 = M(\sigma), \tag{3.23}$$

$M(\sigma)$ being derived from $M(\xi)$ quite simply.

The solution of (3.22) will be considered first. From the matching, the condition that $F_0(0) = 0$ must be imposed (see §4). Also, as F_0 is proportional to the leading term for the velocity on the line of symmetry, we must impose the further condition that F_0 cannot have a singularity at finite σ. Under these conditions, the solution of (3.22) is given by

$$F_0(\sigma) = 2^{-\frac{1}{3}}\{\mathrm{Ai}\,(r)\}^2/[\{\mathrm{Ai}'\,(r)\}^2 - r\{\mathrm{Ai}\,(r)\}^2], \tag{3.24}$$

where Ai is the Airy function, r is given by $r = 2^{-\frac{1}{3}}(\sigma + k)$, $k_* = 2^{-\frac{1}{3}}k$ is any zero of the Airy function, and the dash associated with the Airy function denotes differentiation with respect to r. For a derivation of this solution the reader is referred to Clarke (1966). The solution, as given by (3.24), is displayed graphically in figure 2. For this particular problem the oscillatory behaviour is unrealistic, and so, to exclude it from the region of interest, we choose k_* to be the zero with the smallest magnitude. This corresponds to choosing $k = k_0 = -2{\cdot}94583\ldots$.

For small values of σ, F_0 has the form

$$F_0(\sigma) \sim \tfrac{1}{2}\sigma^2 + \tfrac{1}{12}k_0\sigma^4 + \tfrac{1}{8}\sigma^5 + \ldots. \tag{3.25}$$

It is from the second term of this expression that, on matching, we can assert the form the inner expansion must take. That is, it is the term in ξ^4 of $\hat{q}_{00}$ which forces the existence of the term $R^{\frac{2}{3}}\mathbf{u}_2$ of the inner expansion.

For large values of σ, we find that

$$\hat{q}_{00} \sim \sqrt{(2\xi)} + O(\xi^{-\frac{1}{2}}), \tag{3.26}$$

that is, on returning to dimensional variables, $U \sim (2gX)^{\frac{1}{2}}$, showing that the fluid particles do indeed fall ultimately as a solid body.

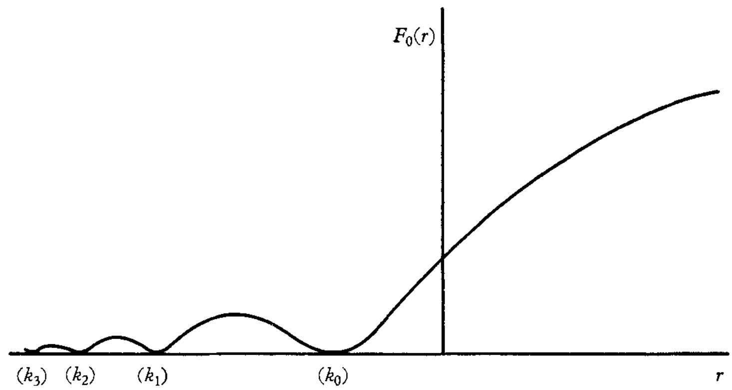

FIGURE 2. Qualitative description of $F_0(r)$ showing oscillatory behaviour.

We turn now to equation (3.23). Because of the complexity of the inhomogeneous part of the equation, the problem of obtaining a solution in closed form was found to be intractable. The associated homogeneous equation can readily be shown to admit the linearly independent solutions

$$v_1 = F_0'(\sigma) \quad \text{and} \quad v_2 = \mathrm{Ai}'\, F_0^2/\mathrm{Ai}^3,$$

and so a Green's function $G(\sigma, \tau)$ may be constructed from v_1 and v_2 in the usua manner. The general solution of (3.23) may then be written as

$$F_2(\sigma) = \int_0^\infty G(\sigma,\tau)\, M(\tau)\, d\tau + \alpha_1 v_1(\sigma) + \alpha_2 v_2(\sigma). \tag{3.27}$$

$v_2(\sigma)$ is exponentially large for $\sigma \to \infty$, and so we may put $\alpha_2 = 0$. From this formal solution, the asymptotic forms for $\sigma \to 0$ and $\sigma \to \infty$ may be obtained,

$$F_2(\sigma) = \tfrac{21}{80}\sigma^{-4} + O(\sigma^{-1}) + \alpha_1(\sigma + O(\sigma^3)), \tag{3.28}$$

for small σ, and for large σ we have

$$F_2(\sigma) = -\tfrac{1}{24}(\sigma + k_0)^{-1} + O(\sigma + k_0)^{-\frac{5}{2}} + \alpha_1(2(\sigma + k_0))^{-\frac{1}{2}} + O(\sigma + k_0)^{-2}. \tag{3.29}$$

We should note here that the outer expansion has been developed under the assumption that the Reynolds number was small, and under such an assumption we may, in principle, obtain an explicit asymptotic representation by appealing to the matching principle. Having obtained this representation, suppose we now allow the Reynolds number to increase until it can no longer be considered small. An examination of the asymptotic forms for F_0 and F_2 (i.e. equations (3.26) and (3.29)) reveals that our representation for $q_0(\xi; R)$ still retains the *appearance* of an asymptotic expansion, though now for $\xi \to \infty$ rather than $R \to 0$. This means that our solution *appears* to be still valid, even for Reynolds numbers which are not small, although the region of validity will be even more restricted than for the case of small Reynolds number.

4. The matching procedure

We are relying on the matching to provide conditions for the equations for q_{0n} of the outer expansion, and also to show the necessity of the existence of terms such as $\mathbf{u}_2$ in the inner expansion. We have seen from the outer expansion that a knowledge of the velocity on the line of symmetry uniquely determines the flow field in the outer region. Hence we need only match the two expansions on that line.

We consider the limiting process $R \to 0$ for $x = M(R)x_M$, with x_M fixed and $1 \ll M(R) \ll R^{-\frac{1}{3}}$. x_M is called an intermediate variable as $x = M(R)x_M \to \infty$ with x_M fixed and $R \to 0$, and $\sigma = 2^{-\frac{4}{3}}R^{\frac{1}{3}}M(R)x_M \to 0$ with x_M fixed and $R \to 0$. We now express the inner and outer expansions in terms of the intermediate variable, the former for $x \to \infty$ and the latter for $\sigma \to 0$, and then compare the two resulting expansions. From §2, we have that the inner expansion, for $x \to \infty$, is

$$\begin{aligned} u(x,0) = {} & \{\tfrac{1}{8}M^2(R)x_M^2 + \tfrac{84}{5}M^{-4}(R)x_M^{-4} + O(M^{-10})\} \\ & + R^{\frac{2}{3}}\{2a_2 M^4(R)x_M^4 + O(M^{-2})\} \\ & + R\{2^{-9}M^5(R)x_M^5 + O(M^{-1})\} \\ & + O(R^{\frac{4}{3}}M^6). \end{aligned} \tag{4.1}$$

As yet we do not know the form for u as given by the outer expansion because we do not know the boundary condition to be applied at $\sigma = 0$. However, we now assert that it must have the same leading term as that provided by the inner expansion. In terms of the intermediate variable, the outer expansion has a leading term $u = \frac{1}{8}M^2(R)x_M^2$. Rephrasing this in the outer variables, we have that

$$F_0(\sigma) \sim \tfrac{1}{2}\sigma^2 \quad \text{as} \quad \sigma \to 0,$$

and hence we have the boundary condition for F_0 (as anticipated in §3.3), $F_0(0) = 0$. On using this condition to solve the equation for $F_0(\sigma)$ we find that for small σ (equation (3.25))

$$F_0(\sigma) = \tfrac{1}{2}\sigma^2 + \tfrac{1}{12}k_0\sigma^4 + \tfrac{1}{8}\sigma^5 + \dots. \tag{4.2}$$

From the terms in (4.1) that were neglected we find that, for matching to one term, the overlap domain is defined by

$$x = M(R)x_M, \quad 1 \ll M(R) \ll R^{-\frac{1}{3}}, \quad (0 < x_M < \infty).$$

We may now express the outer expansion in terms of the intermediate variable,

$$\begin{aligned} u(x,0) = {} & \{\tfrac{1}{8}M^2(R)x_M^2 + \tfrac{1}{3}2^{-\frac{20}{3}}k_0 R^{\frac{2}{3}}M^4(R)x_M^4 + 2^{-9}RM^5(R)x_M^5 + \dots\} \\ & + \{\tfrac{84}{5}M^{-4}(R)x_M^{-4} + O(RM^{-1}) + \alpha_1 O(R^{\frac{5}{3}}M)\} + \dots. \end{aligned} \tag{4.3}$$

The terms on the second line arise from $F_2(\sigma)$. On comparing (4.3) with (4.1) it is seen as expected that the leading terms in each expansion are identical. The next largest terms will be those of order $M^{-4}(R)$ in $1 \ll M(R) \ll R^{-\frac{1}{12}}$ and those of order $R^{\frac{2}{3}}M^4(R)$ in $R^{-\frac{1}{12}} \ll M(R) \ll R^{-\frac{1}{3}}$. However, no attempt is made to distinguish between the relative magnitudes of these two terms, but the assertion is made that both terms are greater than all other terms, other than those already matched. As the forms in x_M of these two terms are so different, no difficulty is encountered in matching them to the two corresponding terms in the other expansion. On matching these terms we find that those of order $M^{-4}(R)$ match automatically, and from those of order $R^{\frac{2}{3}}M^4(R)$ we find $a_2 = \frac{1}{3}2^{-\frac{23}{3}}k_0$, and that the overlap domain for matching to this order is defined by

$$x = M(R)x_M, \quad R^{-\frac{1}{21}} \ll M(R) \ll R^{-\frac{1}{9}}, \quad 0 < x_M < \infty.$$

It is from the matching at this order that we are able to justify the forms (3.15), and also the comments and forms posed in §3.3.

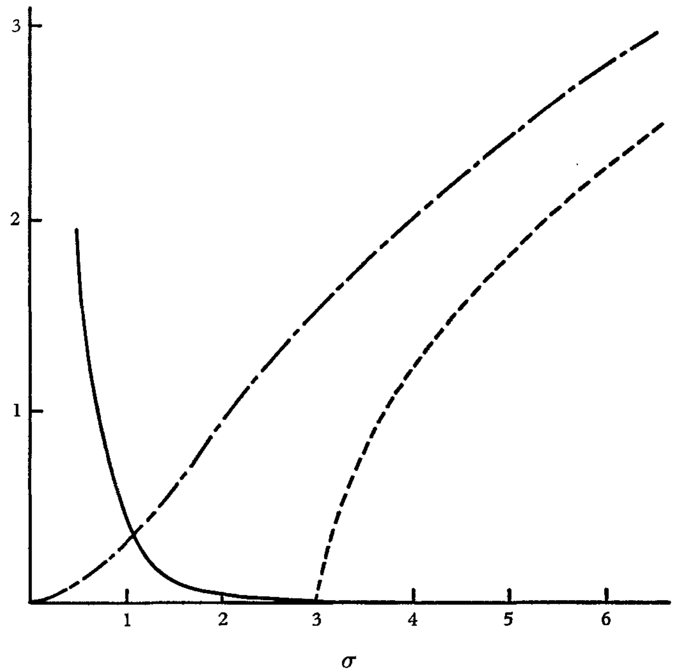

FIGURE 3. Comparison between $F_0(\sigma)$, its asymptote $[2(\sigma+k_0)]^{\frac{1}{2}}$ and $G_0(\sigma)$.
— — — —, $F_0(\sigma)$; - - - -, $[2(\sigma+k_0)]^{\frac{1}{2}}$; ———, $G_0(\sigma)$.

We could in principle continue this process but we shall not do so here. It should be noted that to determine the constant α_1 in the expression for F_2 we would require a knowledge of the second term in the expansion for the fifth approximation of the inner expansion.

In figure 3 we show $F_0(\sigma)$ and compare it with $2^{\frac{1}{2}}(\sigma+k_0)^{\frac{1}{2}}$ to which it asymptotes. Also in figure 3 we show $G_0(\sigma)$, which is essentially $\hat{q}_{20}$, normalized for comparison purposes by $\hat{q}_{20} = 2^{\frac{2}{3}}G_0(\sigma)$ and then expressed in terms of known functions as follows. From equations (3.17) and (3.19) we have that

$$\hat{q}_{20} = \tfrac{1}{2}\hat{q}''_{00}/\hat{q}^2_{00} - \hat{q}'^2_{00}/\hat{q}^3_{00},$$

and so

$$G_0(\sigma) = \tfrac{1}{32}(F''_0/F^2_0 - 2F'^2_0/F^3_0).$$

From figure 3, we can see that G_0, which represents the leading term associated with the variation of the velocity across the jet, has become insignificant, as compared with F_0, for values of σ greater than $\sigma = 3$. This shows that, even for substantial values of R, the velocity distribution across the jet will rapidly tend to become uniform. It should be noted, however, that our solution shows the decay to a uniform distribution to be algebraic. This must be due to the algebraically decaying production of vorticity at the free streamlines, which is in turn due, in this case, to the effects of gravity in accelerating the fluid within the jet.

The fact that G_0, and other higher-order terms in the outer expansion, are singular at the origin ($\sigma = 0$) is not unexpected, it merely reflects that the outer expansion, as well as the inner expansion, is derived from a singular perturbation.

In the preparation of figure 3, considerable use was made of tables of Airy functions by Miller (1946).

Part of this work was done while the author was at the Department of Applied Mathematics and Theoretical Physics in the University of Cambridge. I am grateful to Mr L. E. Fraenkel for his patient guidance during the course of this investigation, as I am to the Science Research Council for a grant during the period of study.

Appendix A. Complex variable formalism of the Navier–Stokes equations in two dimensions

Consider the Navier–Stokes equations written in dyadic notation

$$\rho \mathbf{u} \,.\, \mathrm{grad}\, \mathbf{u} + \rho\, \mathrm{grad}\, \Omega = \mathrm{div}\, \mathsf{T}, \tag{A 1}$$

where Ω is the body force potential, ρ the constant uniform density and T is the stress tensor. We use the equation of continuity

$$\mathrm{div}\, \mathbf{u} = 0, \tag{A 2}$$

to put (A 1) in the form

$$\mathrm{div}\, (\rho \mathbf{u}\mathbf{u} + \rho\Omega \mathsf{I} - \mathsf{T}) = 0, \tag{A 3}$$

where I is the identity tensor. Each term within the bracket in equation (A 3) is a symmetric second-order tensor, and so $(\rho \mathbf{u}\mathbf{u} + \rho\Omega\mathsf{I} - \mathsf{T})$ is also a symmetric second-order tensor, which by (A 3) is divergence free. Equation (A 3) is then the necessary and sufficient condition for the existence of a function $\Phi(x, y)$ such that

$$\frac{\partial^2}{\partial x^2} \Phi = -\rho v^2 - \rho\Omega + \mathsf{T}_{22}, \tag{A 4}$$

$$\frac{\partial^2}{\partial x \partial y} \Phi = \rho uv - \mathsf{T}_{12}, \tag{A 5}$$

$$\frac{\partial^2}{\partial y^2} \Phi = -\rho u^2 - \rho\Omega + \mathsf{T}_{11}, \tag{A 6}$$

(see Muskhelishvili 1963, p. 104).

Φ is an Airy stress function. It can be seen from (A 4) and (A 6) that

$$\nabla^2\Phi = -2\rho(\tfrac{1}{2}q^2+p/\rho+\Omega) = -2\rho H,$$

where H is the total head. If we introduce the well-known expressions for the components of the stress tensor in terms of the pressure and velocity gradients, and use the stream function Ψ, defined by

$$\frac{\partial\Psi}{\partial y} = u \quad\text{and}\quad \frac{\partial\Psi}{\partial x} = -v,$$

equation (A 5) and the difference of (A 4) and (A 6) may be written in the forms

$$\frac{\partial^2}{\partial x\partial y}\Phi = \mu\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\Psi+\rho uv, \tag{A 7}$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\Phi = -4\mu\frac{\partial^2}{\partial x\partial y}\Psi+\rho(u^2-v^2). \tag{A 8}$$

We now change to new independent variables $z(=x+iy)$ and $\bar{z}$ $(=x-iy)$, equations (A 7) and (A 8) becoming

$$\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial \bar{z}^2}\right)\Phi = -i2\mu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \bar{z}^2}\right)\Psi-i\rho uv, \tag{A 7$'$}$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \bar{z}^2}\right)\Phi = -i2\mu\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial \bar{z}^2}\right)\Psi+\tfrac{1}{2}\rho(u^2-v^2). \tag{A 8$'$}$$

The sum of these equations is

$$\frac{\partial^2}{\partial z^2}(\Phi+i2\mu\Psi) = -\rho\left(\frac{\partial\Psi}{\partial z}\right)^2 = \tfrac{1}{4}\rho(u-iv)^2. \tag{A 9}$$

The boundary conditions must also be expressed in terms of Φ and Ψ. On a solid boundary, the velocities will be prescribed, i.e. $\partial\Psi/\partial z$ will be some given function on the boundary contour. On a free streamline the conditions to be applied are that the shear stress, the normal stress and the normal velocity are all to vanish in the case being considered in this paper. The stress conditions may be expressed as $\mathbf{T}.\mathbf{n} = 0$ on the free streamline, where $\mathbf{n}$ is the unit vector normal to the streamline. By use of (A 4)–(A 6), these stress conditions may be put into differential forms

$$\left(\frac{\partial^2\Phi}{\partial y^2}+\rho\Omega+\rho u^2\right)dy+\left(\frac{\partial^2}{\partial x\partial y}\Phi-\rho uv\right)dx = 0,$$

$$\left(\frac{\partial^2\Phi}{\partial x\partial y}-\rho uv\right)dy+\left(\frac{\partial^2}{\partial x^2}\Phi+\rho\Omega+\rho v^2\right)dx = 0,$$

as $\mathbf{n}$ has components proportional to $(dy, -dx)$ where (dx, dy) are the components of a small displacement along the free streamline. We group like terms together and use the identity

$$d\Psi = u\,dy-v\,dx,$$

to find

$$\left.\begin{aligned} d\left(\frac{\partial\Phi}{\partial y}\right)+\rho\Omega\,dy-\rho u\,d\Psi &= 0,\\ d\left(\frac{\partial\Phi}{\partial x}\right)+\rho\Omega\,dx-\rho v\,d\Psi &= 0,\end{aligned}\right\} \tag{A 10}$$

on a free streamline. As $d\Psi = 0$ along any streamline, equations (A 10) may be combined to form one complex condition

$$\frac{\partial \Phi}{\partial \bar{z}} = -\tfrac{1}{2}\rho \int \Omega \, dz, \tag{A 11}$$

on a free streamline.

The condition that the normal velocity should vanish is that $d\Psi = 0$. Considering Ψ as a function of z and $\bar{z}$ this becomes

$$d\Psi = \frac{\partial \Psi}{\partial z} dz + \frac{\partial \Psi}{\partial \bar{z}} d\bar{z}.$$

We therefore have that on a free streamline

$$\mathrm{Re}\left\{\frac{\partial \Psi}{\partial z} dz\right\} = 0. \tag{A 12}$$

Appendix B. Extension to the case of the axially-symmetric jet

The volume flux crossing any section of the jet is defined to be $2\pi Q$. Here Q has different dimensions to the corresponding flux in the two-dimensional case, and so the inner variables will be made dimensionless in a different manner. The dimensionless inner variables are defined by

$$(r_*, z_*) = (Q\nu g^{-1})^{\frac{1}{4}}(r, z), \quad (u_*, w_*) = (Qg\nu^{-1})^{\frac{1}{2}}(u, w), \quad P_* = \rho g (Q\nu g^{-1})^{\frac{1}{4}} p,$$

where (r_*, z_*) are the dimensional cylindrical co-ordinates (the angular component being omitted in view of symmetry), r_* being the radial measure and z_* measuring along the axis of symmetry.

From the equation of continuity, there exists a dimensional stream function Ψ given by $r_* w_* = \partial\Psi/\partial r_*$, $-r_* u_* = \partial\Psi/\partial z_*$. Ψ is rendered dimensionless by $\Psi = Q\psi$. This means that the line of symmetry is denoted by $\psi = 0$ and the free surface by $\psi = 1$. The only dimensionless parameter appearing is again a Reynolds number, now defined by $R = g^{\frac{1}{3}} Q \nu^{-\frac{5}{3}}$.

The complex variable method is now unavailable, and no attempt will be made here to solve the problem in the inner region. However, it can be seen that the solution to the Stokes equation will again have an asymptotic form, for large z, $w \sim O(z^2)$. This is sufficient to provide the boundary condition for the first approximation in the outer region.

The outer variables are defined by $(r_*, z_*) = (\nu^2 g^{-1})^{\frac{1}{3}}(\hat{r}, \hat{z})$, $(u_*, w_*) = (\nu g)^{\frac{1}{3}}$ $(\hat{u}, \hat{w})$, $P_* = \rho(\nu g)^{\frac{2}{3}}\hat{p}$ and $\Psi = R\nu^{\frac{5}{3}} g^{-\frac{1}{3}}\psi$. The free surface is again denoted by $\psi = 1$. The relationship between the inner and outer variables is $(r, z) = R^{-\frac{1}{4}}$ $(\hat{r}, \hat{z})$, $(u, w) = R^{-\frac{1}{2}}(\hat{u}, \hat{w})$ and $p = R^{-\frac{1}{4}}\hat{p}$. The circumflex will now be omitted for convenience, as only outer variables will be used from now on. As in the two-dimensional case, the problem is transformed onto the ζ-space, where $\zeta = (\xi, R\psi, \eta)$, η being the angular co-ordinate. We again define ξ by $\xi = z$ on the line of symmetry and the surfaces $\xi = $ constant are everywhere orthogonal to the surfaces $R\psi = $ constant.

An angle θ is defined by $(u, w) = (-q\sin\theta, q\cos\theta)$, with $q^2 = v^2+w^2$. Consider an annular element lying in the surface ξ = constant. The volume flux through this element is given by $2\pi Q d\psi$; it is also given by $2\pi\, \nu^{\frac{5}{3}} g^{-\frac{1}{3}} rq\, ds$, where ds is the width of the element. Therefore $ds = R\, d\psi/(rq)$ and hence the two sets of co-ordinates are linked by

$$z = \xi + R\int_0^{\psi} \sin\theta r^{-1} q^{-1}\, \mathrm{d}\psi, \tag{B 1}$$

$$r = R\int_0^{\psi} \cos\theta\, r^{-1} q^{-1}\, \mathrm{d}\psi. \tag{B 2}$$

The factor r^{-1} may be removed from within the integral in (B 2) to clarify the form of r. We differentiate (B 2) with respect to ψ and multiply both sides of the equation by r to obtain

$$r\frac{\partial r}{\partial \psi} = Rq^{-1}\cos\theta.$$

We integrate this equation with respect to ψ from 0 to ψ to obtain

$$r^2 = 2R\int_0^{\psi} q^{-1}\cos\theta\, d\psi. \tag{B 3}$$

The arc-length parameters associated with $(\xi, R\psi, \eta)$ are $(h, q^{-1}r^{-1}, r)$, h again being unknown. The conditions for the irrotationality and divergence-free nature of a constant vector provide the equations

$$\frac{\partial}{\partial \xi}\left(\frac{\sin\theta}{rq}\right) = \frac{1}{R}\frac{\partial}{\partial \psi}(h\cos\theta), \tag{B 4}$$

$$\frac{\partial}{\partial \xi}\left(\frac{\cos\theta}{q}\right) = -\frac{1}{R}\frac{\partial}{\partial \psi}(hr\sin\theta). \tag{B 5}$$

The Navier–Stokes equations become in the new co-ordinate system

$$q\frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} = h\cos\theta + R^{-2}hq\frac{\partial}{\partial \psi}\left\{\frac{r^2 q}{h}\frac{\partial}{\partial \psi}(qh)\right\}, \tag{B 6}$$

$$-\frac{q^2}{h}\frac{\partial h}{\partial \psi} + \frac{\partial p}{\partial \psi} = \frac{R\sin\theta}{qr} - \frac{1}{hqr^2}\frac{\partial}{\partial \xi}\left\{\frac{r^2 q}{h}\frac{\partial}{\partial \psi}(qh)\right\}, \tag{B 7}$$

and the boundary conditions to be applied on $\psi = 1$ are

$$\frac{\partial}{\partial \psi}(q/h) = 0, \tag{B 8}$$

$$p = \frac{2qr}{h}\frac{\partial}{\partial \xi}\left(\frac{1}{qr}\right). \tag{B 9}$$

Equations (B 3), (B 4) and (B 5) suggest that the dependent variables may be expanded in the forms

$$\left.\begin{aligned} q &= q_0 + R\psi q_1 + \dots, \\ p &= p_0 + R\psi p_1 + \dots, \\ h &= 1 + R\psi h_1 + \dots, \\ \theta &= (R\psi)^{\frac{1}{2}}(\theta_0 + R\psi\theta_1 + \dots), \\ r &= (R\psi)^{\frac{1}{2}}((\tfrac{1}{2}q_0)^{-\frac{1}{2}} + R\psi r_1 + \dots). \end{aligned}\right\} \tag{B 10}$$

Again the coefficients of $R\psi$ will in general be functions of both ξ and R, with the leading term independent of R, i.e.

$$q_0(\xi, R) = q_{00}(\xi) + R^\nu q_{01}(\xi) + \ldots, \quad (\nu > 0).$$

We proceed as for the two-dimensional case to find that the equation for q_{00} is

$$3q''_{00} - 3q'^2_{00}/q_{00} - q_{00}q'_{00} + 1 = 0. \tag{B 11}$$

We put $\xi = 3^{\frac{2}{3}}\sigma$ and $q_{00} = 3^{\frac{1}{3}}F_0(\sigma)$ and the equation (B 11) reduces to

$$F''_0 - F'^2_0/F_0 - F_0F'_0 + 1 = 0, \tag{B 12}$$

which is the same equation as the one for the leading term in the two-dimensional case, and hence the solution is known. If we expand the solution for small values of σ and use the matching principle, it is easily seen that the inner expansion will be one in powers of $R^{\frac{1}{3}}$.

REFERENCES

BROWN, D. R. 1961 A study of the behaviour of a thin sheet of moving liquid. *J. Fluid Mech.* **10**, 297.

CLARKE, N. S. 1966 A differential equation in fluid mechanics. *Mathematika*, **12**, 51.

GARABEDIAN, P. R. 1966 Free boundary flows of a viscous liquid. *Comm. Pure Appl. Math.* **19**, no. 4, 421.

LANGLOIS, W. E. 1964 *Slow Viscous Flow*. New York: Macmillan.

LEGENDRE, R. 1949 Laminar flow along a flat plate. *Comptes Rendus*, June, **228**, part 2.

MILLER, J. C. P. 1946 *The Airy Integral*. British Association for the Advancement of Science, Mathematical Tables, part-vol. B, Cambridge University Press.

MOISIL, Gr. C. 1955 Comunicarile Academiei Republicii Populare Romine. *Tomul* V, nr. 10.

MUSKHELISHVILI, N. I. 1963 *Some Basic Problems of the Mathematical Theory of Elasticity*. Groningen: P. Noordhoff.